

Magnetic susceptibility at zero and nonzero chemical potential in QCD and QED

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Abstract

Magnetic susceptibility of the quark matter in QCD is calculated in a closed form for an arbitrary chemical potential μ . For small μ , $\mu \ll T$, $\sqrt{eB} \ll T$, a strong dependence on temperature T is found due to Polyakov line factors. In the opposite case of small T , $\sqrt{eB} \gtrsim T$, the oscillations as functions of eB occur, characteristic of the de Haas-van Alphen effect. Results are compared with available lattice data.

1 Introduction

The important role of magnetic fields (m.f) in nature has recently become a topic of a vivid interest. Strong m.f. are expected in cosmology [1] and in astrophysics (magnetars) [2], very large m.f. can occur in heavy-ion collisions [3], where a temperature transition to the quark-gluon matter is expected. For a modern review of these topics see [4].

For a theory the m.f. effects play an additional role of a crucial test, which proves or disproves the assumed intrinsic dynamics, or shows its boundaries.

Of special importance strong m.f. are in QCD, since both hadrons and the quark matter are possible parts of neutron stars and m.f. can occur in heavy ion collisions.

Recently a new approach was suggested to treat QCD and QED in m.f., based on the relativistic Hamiltonians, deduced from the QCD path integral [5, 6, 7].

A striking result of this approach is the strong reduction of the hadron masses due to m.f. in mesons [8, 9, 10], and in baryons [11]. For the neutron the mass is twice as small for $eB = 0.2 \text{ GeV}^2$. These results for mesons are supported by lattice data [12, 13, 14].

Another feature of m.f. is the lowering of the temperature T_c of the transition from the hadronic to the quark-gluon matter, which was found in the same path integral approach [15, 16] and supported by the lattice data [17]. It is a purpose of the present paper to develop the theory further and to find in a simple closed form the magnetic susceptibility (m.s.) of the quark matter for an arbitrary chemical potential μ . Recently this type of analysis was done for the zero μ [18] and the numerical results for m.s. have been compared to lattice data [19, 20, 21, 22], showing a good agreement.

The nonzero μ case is interesting from several points of view. First of all it covers the regions of μ, T, eB which are present also in the case of the electron gas, and where the effects of the Pauli paramagnetism [23] and Landau diamagnetism [24] occur, moreover there also the de Haas-van Alphen phenomenon is possible, see the regular course [25] for a general discussion. As we shall see, in the case of the quark gas a simple modification occurs in all these effects, and in addition another, and possibly more important region, $\mu \ll T$ exists, where our method allows to obtain simple general results.

The paper is organized as follows. In the next section general expressions for the thermodynamical potentials in m.f. are derived, in section 3 the expression for the m.s. $\hat{\chi}$ is deduced. Section 4 is devoted to the m.s. at nonzero μ , while in the section 5 the classical Pauli, Landau and de Haas-van Alphen effects are demonstrated for the quark matter.

In section 6 the main results are summarized and perspectives are given.

2 A general theory of the fermion gas in magnetic field

We start with the case of the electron gas in m.f., where the thermodynamical potential $\Omega(V, T, \mu)$ (or rather $\Omega(T) - \Omega(0) \equiv -P$), which we use in what follows, can be written as [25]

$$P_e(B, \mu, T) = \sum_{n_\perp, \sigma} \frac{eBT}{2\pi} X(\mu), \quad X(\mu) = \int \frac{dp_z}{2\pi} \ln \left(1 + \exp \left(\frac{\mu - E_{n_\perp}^\sigma(B)}{T} \right) \right), \quad (1)$$

where

$$E_{n_\perp}^\sigma(B) = \sqrt{p_z^2 + (2n_\perp + 1 - \sigma)eB + m_e^2}, \quad \sigma = \pm 1. \quad (2)$$

Note, that (2) is the relativistic generalization of the standard expression [25] in the theory of the electron (or electron-positron) gas in m.f. at nonzero temperature. It was a subject of an intensive study during the last 50 years, see e.g. [26, 27, 28, 29, 30]. The QED relativistic thermodynamical potential in m.f. at finite T and density was obtained in [26], and using the generalized Fock-Schwinger method for $\mu \neq 0, T \neq 0$ in [27]. In the case of the zero temperature and nonzero μ, B the useful form of the effective action was obtained in [28], and finally the full expression for nonzero, μ, T, B was presented in [29]. Simple forms of effective Lagrangians for $T = 0$ and oscillations as function of m.f. are obtained in [30]. For further developments and discussions and limiting cases see also [31]. These results have been exploited and augmented by the study of the quark-antiquark gas also in magnetic field [32, 33, 34].

In the latter case one can write for a given sort of quarks and antiquarks similarly to (1), if one neglects the effect of the vacuum QCD fields on quarks

$$P_q(B, \mu, T) = \sum \frac{N_c e_q B T}{2\pi} (X_q(\mu) + X_q(-\mu)), \quad (3)$$

and $X_q(\mu)$ has the same form as in (1), (2) with $e = e_q \equiv |e_q|$, and $m_e \rightarrow m_q$.

However, the vacuum QCD fields, which are responsible for confinement at $T < T_c$ [35], also affect the quark gas. The theory of both confined and deconfined matter was suggested in [36] and finally formulated, basing on the path integral formalism and the Field Correlator Method (FCM) in [37, 38, 39], for a review see [40].

In this formalism, neglecting the $q\bar{q}$ weakly bound states around T_c (the “Single Line Approximation” SLA [37]) one arrives at the simple modification of the expression (3), where one should replace in $X(\mu)$ the chemical potential μ as follows

$$\exp \frac{\mu}{T} \rightarrow \exp \frac{\mu_q}{T} L(T), \quad (4)$$

where $L(T) = \exp\left(-\frac{V_1(\infty, T)}{2T}\right)$ is the average value of the fundamental Polyakov line, which was studied analytically in [39, 40] and numerically on the lattice in [41].

As a result of integration over dp_z in $X(\mu)$ one arrives at the expression [15], containing a sum over Matsubara numbers

$$P_q(B, \mu, T) = \frac{N_c e_q B T}{\pi^2} \sum_{n_\perp, \sigma} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} L^n \frac{e^{\frac{n\mu}{T}} + e^{-\frac{\mu n}{T}}}{T} \varepsilon_{n_\perp}^\sigma K_1\left(\frac{n \varepsilon_{n_\perp}^\sigma}{T}\right) \quad (5)$$

with $K_1(z)$ – the modified Bessel function and

$$\varepsilon_{n_\perp}^\sigma = \sqrt{e_q B (2n_\perp + 1 - \sigma) + m_q^2}. \quad (6)$$

Another form of (5) was obtained in [15] by direct summing $\sum_{n_\perp, \sigma} X_q(\mu)$ in (3), which gives the integral expressions

$$P_q(B, \mu, T) = \frac{N_c e_q B}{2\pi^2} (\psi(\mu) + \psi(-\mu)), \quad \psi(\mu) = \phi(\mu) + \frac{2}{3} \frac{\lambda(\mu)}{e_q B} - \frac{e_q B \tau(\mu)}{24}, \quad (7)$$

where $\phi(\mu)$, $\lambda(\mu)$ and $\tau(\mu)$ are integrals over momenta given in (35), (36), (37).

In what follows we shall be mostly using the form (5), which was summed up over n_\perp, σ for $\mu = 0$ in [15]

$$P_q(B, \mu, T) = \frac{N_c e_q B T}{\pi^2} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} L^n \left\{ m_q K_1\left(\frac{n m_q}{T}\right) + \right. \\ \left. + \frac{2T}{n} \frac{e_q B + m_q^2}{e_q B} K_2\left(\frac{n}{T} \sqrt{e_q B + m_q^2}\right) - \frac{n e_q B}{12T} K_0\left(\frac{n}{T} \sqrt{m_q^2 + e_q B}\right) \right\}. \quad (8)$$

It is easy to see, that the case of $\mu > 0$ obtains by the formal replacement

$$L^n \rightarrow L^n \text{ch}\left(\frac{\mu n}{T}\right) = \frac{L_\mu^n + L_{-\mu}^n}{2}, \quad L_\mu \equiv e^{\frac{\mu}{T}} L. \quad (9)$$

The form (8) or its nonzero μ equivalent (9) have a nice property of yielding correct limiting values for 1) $e_q B \rightarrow 0$, 2) $e_q B \rightarrow \infty$, 3) $T \gg m_q, e_q B$.

In the first case only the second term inside curly brackets in (8) contributes and one has

$$P_q(0, \mu, T) = \frac{N_c 2T^2 m_q^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^2} \frac{L_\mu^n + L_{-\mu}^n}{2} K_2\left(\frac{n m_q}{T}\right). \quad (10)$$

In the second case only the first term inside curly brackets survives and we obtain

$$P_q(B, \mu, T)|_{B \rightarrow \infty} = \frac{N_c e_q B T m_q}{\pi^2} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \frac{L_\mu^n + L_{-\mu}^n}{2} K_1\left(\frac{n m_q}{T}\right). \quad (11)$$

At large T , $T \gg \sqrt{m_q^2 + e_q B}$, the leading term in (8) is again the second in the curly brackets and one has

$$P_q(B, \mu, T \rightarrow \infty) = \frac{4 N_c T^4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^4} \frac{L_\mu^n + L_{-\mu}^n}{2}, \quad (12)$$

which yields for $\mu = 0, L = 1$ the standard result

$$P_q(B, \mu = 0, T \rightarrow \infty) = \frac{7\pi^2 N_c T^4}{180}, \quad \bar{P}_q = \sum_q P_q = \frac{7\pi^2 N_c T^4}{180} n_f. \quad (13)$$

At this point one should stress the importance of the explicit summation over n , especially when $\mu \neq 0$. This indeed can be done as in [15] with the result given in (7).

Finally we should comment on the accuracy of our representation (8), (9), which is obtained, when the summation over n_\perp with $\sigma = -1$ in (5) is performed, using the Euler-Mc Laurent approximation (see § 59 of [25] for a discussion)

$$\sum_{n_\perp=0}^{\infty} F(n_\perp + \frac{1}{2}) \cong \int_0^\infty F(x) dx + \frac{1}{24} F'(0), \quad (14)$$

which yields the first term in the curly brackets in (8), (9). This substantiates the good accuracy of the total expression in the whole region of parameters except for a narrow region $T \ll m_q$, $T \lesssim \frac{e_q B}{2m_q} \ll \mu - m_q \equiv \mu_0$ where an oscillating regime sets in, considered in the next sections. As an additional check of this accuracy we show in the next section that in the expansion of P_q in powers of $(e_q B)^k$ the terms with $k = 1$ and 3 vanish identically.

One finds that (14) is accurate within the terms $O\left(\frac{\sqrt{m_q^2 + e_q B}}{T}\right)$, when $\sqrt{m_q^2 + e_q B} \ll T$, while in the opposite case the sum (14) is much smaller than the term with $\sigma = 1$.

3 Magnetic susceptibility of the quark matter

In this section we are specifically interested in the $e_q B$ dependence of $P_q(B, \mu, T)$ and first of all in the quadratic term of this expansion – the magnetic susceptibility (m.s.). To this end we are exploiting the integral representation of K_n

$$K_\nu(z) = \frac{1}{2} \left(\frac{2}{z}\right)^\nu \int_0^\infty e^{-t - \frac{z^2}{4t}} t^{\nu-1} dt, \quad K_\nu = K_{-\nu}, \quad (15)$$

which allows one to write expansions

$$\frac{2T}{n} (e_q B + m_q^2) K_2 \left(\frac{n \sqrt{e_q B + m_q^2}}{T} \right) = \frac{2T m_q^2}{n} \sum_{k=0}^\infty \left(\frac{e_q B n}{2T m_q} \right)^k \frac{(-)^k}{k!} K_{k-2} \left(\frac{n m_q}{T} \right), \quad (16)$$

$$K_0 \left(\frac{n \sqrt{m_q^2 + e_q B}}{T} \right) = \sum_{k=0}^\infty \left(\frac{n e_q B}{2T m_q} \right)^k \frac{(-)^k}{k!} K_k \left(\frac{n m_q}{T} \right) \quad (17)$$

and as a result Eqs. (8), (9) assume the form

$$P_q(B, \mu, T) - P_q(0, \mu, T) = \frac{(e_q B)^2 N_c}{2\pi^2} \sum_{n=1}^\infty (-)^{n+1} \frac{(L_\mu^n + L_{-\mu}^n)}{2} f_n, \quad (18)$$

$$f_n = \sum_{k=0}^\infty \frac{(-)^k}{k!} \left(\frac{n e_q B}{2T m_q} \right)^k K_k \left(\frac{n m_q}{T} \right) \left[\frac{1}{(k+1)(k+2)} - \frac{1}{6} \right]. \quad (19)$$

Note, that the linear term in (16) exactly cancels the term $m_q K_1 \left(\frac{n m_q}{T} \right)$ in (8), so that the sum in (18) starts with the quadratic term, also the cubic term vanishes in (19). Hence one can define the m.s. $\hat{\chi}_q$

$$P_q(B, \mu, T) - P_q(0, \mu, T) = \frac{\hat{\chi}_q}{2} (e_q B)^2 + O((e_q B)^4). \quad (20)$$

As the result, one arrives at the following expression for $\hat{\chi}_q$

$$\hat{\chi}_q(T, \mu) = \frac{N_c}{3\pi^2} \sum_{n=1}^\infty (-)^{n+1} \frac{L_\mu^n + L_{-\mu}^n}{2} K_0 \left(\frac{n m_q}{T} \right), \quad (21)$$

with $L_\mu \equiv L \exp \frac{\mu}{T}$.

As the next step we are using for $K_0(z)$ the relation

$$K_0\left(\frac{nm_q}{T}\right) = \frac{1}{2} \int_0^\infty \frac{dx}{x} e^{-n\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)} \quad (22)$$

and obtain the final expression, summing over n ,

$$\hat{\chi}_q(T, \mu) = \frac{N_c}{3\pi^2} \frac{J_q(\mu) + J_q(-\mu)}{2}, \quad J_q(\mu) = \frac{1}{2} \int_0^\infty \frac{dx}{x} \frac{L_\mu e^{-\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)}}{1 + L_\mu e^{-\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)}}. \quad (23)$$

The total m.s. $\hat{\chi}(T, \mu)$ for a quark ensemble with n_f species is defined as

$$\hat{\chi}(T, \mu) = \sum_q \hat{\chi}_q(T, \mu) \left(\frac{e_q}{e}\right)^2. \quad (24)$$

Note, that one can define a more general form, appropriate for the comparison with numerical simulations, when one simply extracts the quadratic term $(e_q B)^2$, leaving m.f. nonzero in the rest terms, namely

$$\hat{\chi}_q(B, T, \mu) = \frac{2P_q(B, \mu, T)}{(e_q B)^2} = \frac{N_c}{3\pi^2} \sum_{n=1}^\infty (-)^{n+1} \frac{L_\mu^n + L_{-\mu}^n}{2} \varphi_n(m_q^2 + e_q B) \quad (25)$$

with

$$\varphi_n(m_q^2 + e_q B) = \ln \left(\frac{2T}{n\sqrt{e_q B + m_q^2}} \right) + 0.33. \quad (26)$$

One can see in (25), (26), that m_q^2 enters in $\hat{\chi}_q(B, T, \mu)$ always in combination with $e_q B$, so that one can define an effective mass

$$(m_q^2)_{eff} = m_q^2 + e_q B, \quad (27)$$

and $e_q B$ is of the order of the minimal m.f. present in the lattice measurement of $\hat{\chi}_q$, which is usually larger, than m_u^2, m_d^2 .

Note, however, that the series over n in Eq. (25) is not well convergent for $L_\mu > 1$ and one should do in this case an explicit summation over n yielding (23).

One can see in (21), (25), that at large $T \gg m_q, \sqrt{m_q^2 + e_q B}$, each term in (21), (25) behaves as $\sim \ln \frac{T}{m_q}$, implying that $\chi_{q_1} > \chi_{q_2}$, when $m_{q_2} > m_{q_1}$.

However, summation over n yields in (23) the denominator which flattens the logarithmic growth of the first term in the sum. This situation is especially interesting for the free case, when $L_\mu \equiv 1$ ($\mu = 0$), in which case $\hat{\chi}_q^{(0)} \approx \frac{N_c}{3\pi^2} \sum_n (-1)^{n+1} K_0\left(\frac{nm_q}{T}\right)$ and summation over n leads to the Eq. (23) with $L_\mu = 1$. The numerical result for the first term $\hat{\chi}_q^{(0)}(n=1) = \frac{N_c}{3\pi^2} K_0\left(\frac{m_q}{T}\right)$ and for the whole sum is shown in Fig. 1. The corresponding expression of $\hat{\chi}_q^{(0)}$ for the electron gas, which is twice as small, can be found in [21], and in [29].

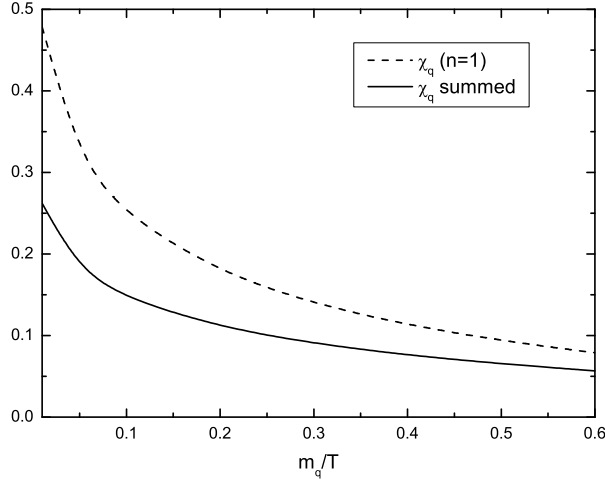


Figure 1: Magnetic susceptibility in SI units ($\chi_q = \frac{4\pi}{137} \hat{\chi}_q$) in free case, $L_\mu = 1$, Eq. (23) (solid line), in comparison with the first term in the sum (21).

As the next step one must define the Polyakov line, which in the neighborhood of T_c was found analytically in [39, 40] as

$$L \equiv L^{(V)}(T) = \exp\left(-\frac{V_1(\infty, T)}{2T}\right), \quad V_1(\infty, T) \approx V_1(\infty, T_c) = 0.5 \text{ GeV}. \quad (28)$$

Note, that by derivation in [37] the Polyakov line $L^{(V)}(T)$ takes into account only the single quark interaction with the vacuum, given by $V_1(\infty, T)$, hence the superscript V , while on the lattice [41] one measures the full Polyakov

line, which can be expressed via the free energy $F_1(\infty, T)$,

$$L^{(F)}(T) = \exp\left(-\frac{F_1(\infty, T)}{2T}\right). \quad (29)$$

As argued in [40], $F_1 < V_1$ and hence $L^{(F)}(T) > L^{(V)}(T)$. In [18] both forms of $L(T)$ have been used for comparison with lattice data for m.s. without chemical potential.

It is interesting to compare $\hat{\chi}_q(T, \mu)$ for three different sorts of quarks, u, d, s . Using (23) with $m_q^2 \rightarrow m_q^2(\text{eff})$, one can find three curves for $m_u(\text{eff}) = 68$ MeV, $m_d(\text{eff}) = 49$ MeV, $m_s(\text{eff}) = 111$ MeV and $\mu = 0$, which are in good agreement with the lattice data from [21], see Fig. 2 (left graph), where we use $L = L^{(V)}$ from (28). The sum of different quarks contributions is shown on Fig. 2, right graph.

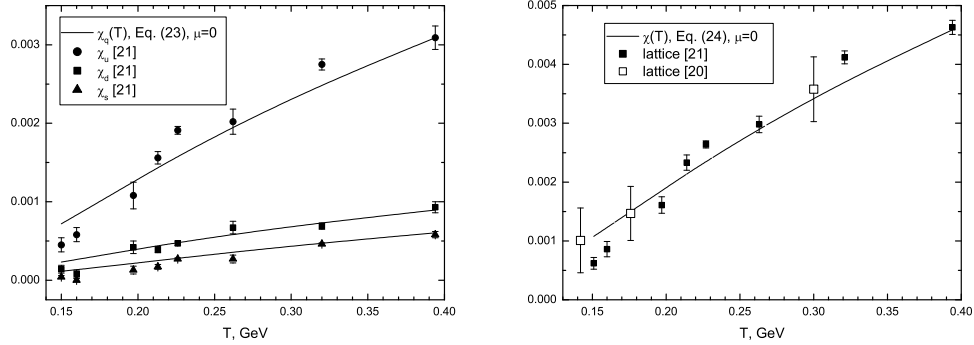


Figure 2: Magnetic susceptibility in SI units ($\chi_q = \frac{4\pi}{137}\hat{\chi}_q$) as a function of temperature for different sorts of quarks (left graph) and the total magnetic susceptibility (right graph) for the case $\mu = 0$ in comparison with lattice data [20, 21].

4 The case of nonzero chemical potential

For $\mu > 0$ one can use the standard representation (cf. Eq. (24) of [15])

$$P_q(B) = N_c T \frac{e_q B}{2\pi} (\psi(\mu) + \psi(-\mu)), \quad (30)$$

where

$$\psi(\mu) = \sum_{n_{\perp}\sigma} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \ln \left(1 + \exp \left(\frac{\bar{\mu} - E_{n_{\perp}}^{\sigma}(B)}{T} \right) \right), \quad (31)$$

and

$$\bar{\mu} = \mu - \bar{J} = \mu - \frac{V_1(\infty, T)}{2}, \quad \bar{L}_{\mu} = \exp \left(\frac{\bar{\mu} - \bar{J}}{T} \right) = \bar{L} \exp \left(\frac{\bar{\mu}}{T} \right), \quad (32)$$

$$E_{n_{\perp}}^{\sigma}(B) = \sqrt{p_z^2 + (2n_{\perp} + 1 - \sigma)e_q B + m_q^2}. \quad (33)$$

Separately out in (31) the term $\sigma = 1, n_{\perp} = 0$ one can rewrite $\psi(\mu)$ as (see Appendix of [15] for details)

$$\psi(\mu) = \frac{1}{\pi T} \left\{ \phi(\mu) + \frac{2}{3} \frac{\lambda(\mu)}{e_q B} - \frac{e_q B}{24} \tau(\mu) \right\}, \quad (34)$$

where $\phi(\mu)$ does not depend on $e_q B$,

$$\phi(\mu) = \int_0^{\infty} \frac{p_z dp_z}{1 + e^{\frac{p_z - \bar{\mu}}{T}}}, \quad (35)$$

$$\lambda(\mu) = \int_0^{\infty} \frac{p^4 dp}{\sqrt{p^2 + \tilde{m}_q^2}} \frac{1}{1 + \exp \left(\frac{\sqrt{p^2 + \tilde{m}_q^2} - \bar{\mu}}{T} \right)}, \quad (36)$$

$$\tau(\mu) = \int_0^{\infty} \frac{dp}{\sqrt{p^2 + \tilde{m}_q^2} \left(1 + \exp \left(\frac{\sqrt{p^2 + \tilde{m}_q^2} - \bar{\mu}}{T} \right) \right)}, \quad (37)$$

and $\tilde{m}_q^2 = m_q^2 + e_q B$.

It is clear, that with 4 dimensionful parameters $\bar{\mu}, m_q, T, e_q B$ one has more than 6 limiting regions. Therefore in this section we shall confine ourselves to only three situations, out of which two were treated in [25] for the electron gas and called there a) the case of weak fields,

$$T \ll \varepsilon_F = \mu_0, \quad T \ll m_q, \quad \frac{eB}{2m_q} \ll T, \quad (38)$$

and b) the case of strong fields,

$$T \lesssim \frac{eB}{2m_q} \ll \mu_0, \quad T \ll m_q, \quad \mu_0. \quad (39)$$

In addition, there is another interesting region, namely $T \gg \mu, e_q B < T^2$, leading some access to the numerical simulations, which will be considered now, while the cases a) and b) are discussed in the next section.

We consider here the case of small $\mu, \mu \ll T$, and small m.f., $\sqrt{eB} \ll T$, when the possible region of oscillations due to the sum over integrals n_\perp in (2) is unimportant, and one can replace the sum by the integral, as it is done in (8), (9) using (14). In this case one can use (23) with L_μ given by (9) and $L(T)$ due to (28), (29). We note here, that the influence of μ on $L(T)$ is expected here to be negligible, see e.g. lattice data in [42].

In Fig. 3 we show a typical behavior of $\hat{\chi}_q(T, \mu)$, given by (23) for $L(T) = L^{(V)}(T)$ from (28). For $m_u = 68$ MeV (as for Fig. 2) and $\mu \equiv \mu_u = (0, 100, 200)$ MeV one can see a set of curves $\hat{\chi}_u$ as a function of T in the interval (150-400) MeV.

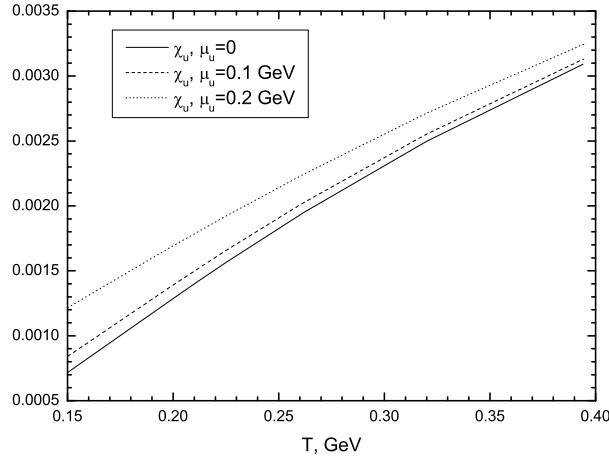


Figure 3: Magnetic susceptibility χ_u from (23) in SI units ($\chi_q = \frac{4\pi}{137}\hat{\chi}_q$) as a function of temperature for nonzero values of chemical potential μ .

Another possible characteristics of the small μ influence is the quark number susceptibility of $\hat{\chi}_q(T, \mu_q)$, given by

$$\hat{\chi}_q^{(\mu)}(T) \equiv \left. \frac{\partial^2 \hat{\chi}_q(T, \mu)}{\partial \mu^2} \right|_{\mu=0} = \frac{N_c T^2}{6\pi^2} \left(\frac{\partial^2 J_q(\mu)}{\partial \mu^2} + \frac{\partial^2 J_q(-\mu)}{\partial \mu^2} \right) \Big|_{\mu=0}. \quad (40)$$

Differentiating (23) one obtains

$$\chi_q^{(\mu)}(T) = \frac{N_c}{12\pi^2} \int_0^\infty \frac{dx}{x} \frac{L_\mu e^{-\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)} \left(1 - L_\mu e^{-\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)}\right)}{\left(1 + L_\mu e^{-\left(\frac{1}{x} + \frac{m_q^2 x}{4T^2}\right)}\right)^3} + (\mu \rightarrow -\mu). \quad (41)$$

This combined quark-number and magnetic susceptibility is a generalization of the powerful technic of the study of the chemical potential influence on thermodynamic potentials and phase transition on the lattice, (see a recent paper [43] for a discussion and references).

One should note, that the corresponding quark number susceptibility (q.n.s.) was calculated for zero m.f. in the framework of our approach in [38]. To this end one can use (30), (36), since only $\lambda(\mu)$ survives for $e_q B \ll m_q$, and one has

$$P_q(B=0) = \frac{N_c}{3\pi^2} \left\{ \lambda(0) + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k \lambda(\mu - q)}{\partial \left(\frac{\mu_q}{T}\right)^k} \left(\frac{\mu_q}{T}\right)^k \right\} \quad (42)$$

and $\lambda(\mu_q)$ is given by (36).

The equivalent series for $\hat{\chi}_q(T, \mu)$ is obtained by the replacement in (42), $\lambda(\mu_q) \rightarrow \frac{1}{2}(J_q(\mu) + J_q(-\mu))$.

5 The cases of strong and weak fields

We consider now the cases of the weak and strong fields, Eqs. (38) and (39) respectively, essentially the material of §§ 58,59 of [25].

Note, that our μ for quarks contains the quark mass, $\mu = m_q + \mu_0$, while μ_0 depends on density, $\mu_0 = \varepsilon_F$ for electron gas.

We start with the case a) $\frac{eB}{2m_q} \ll T \ll m_q, \mu_0$.

Here one can use (30), (34) and take into account, that the quadratic in $e_q B$ terms come only from $\lambda(\mu)$ and $\tau(\mu)$. We neglect the terms $O\left(\frac{e_q B}{m_q^2}\right)$ (and hence $\psi(-\mu)$ in (30)) and omitting $V_1(\infty, T)$ write the exponent in the

integrand of (36) as

$$\exp\left(\frac{\sqrt{p^2 + \tilde{m}_q^2} - \bar{\mu}}{T}\right) = \exp\left(\frac{p^2}{2m_q T} - \frac{\bar{\mu}_0}{T}\right), \quad \bar{\mu} = \mu_0 + m_q,$$

$$\bar{\mu}_0 = \mu_0 - \frac{e_q B}{2m_q} \equiv \mu - m_q - \frac{e_q B}{2m_q}, \quad (43)$$

and the integral (36) can be rewritten, using the variable $z = \frac{p^2}{2m_q T}$,

$$\lambda(\mu) = T^4 \left(\frac{2m_q}{T}\right)^{3/2} \int_0^\infty \frac{z^{3/2} dz}{e^{z - \bar{\mu}_0/T} + 1}. \quad (44)$$

The integral on the r.h.s. of (44) is exactly of the type considered in [25], § 58, where the asymptotic series was obtained in powers $\left(\frac{\bar{\mu}_0}{T}\right)^k$. Keeping the leading term, one has

$$\lambda(\mu) = T^4 \left(\frac{2m_q}{T}\right)^{3/2} \left(\frac{2}{5} \left(\frac{\bar{\mu}_0}{T}\right)^{5/2} + O\left(\frac{\bar{\mu}_0}{T}\right)^{1/2}\right). \quad (45)$$

Expanding $\bar{\mu}_0 = \tilde{\mu}_0 - \frac{e_q B}{2m_q}$, and keeping the term $\left(\frac{e_q B}{2m_q}\right)^2$, one obtains from (30), (34) the paramagnetic contribution to P_q

$$P_q^{(2)} = \frac{N_c (e_q B)^2}{4\pi^2} \left(\frac{\mu_0}{2m_q}\right)^{1/2}, \quad (46)$$

and for $\tau(\mu)$ one has

$$\tau(\mu) = \sqrt{\frac{m_q T}{2}} 2 \left(\frac{\bar{\mu}_0}{T}\right)^{1/2}. \quad (47)$$

Inserting these values into (30), (34) one obtains for the $\tau(\mu)$ the contribution to $\hat{\chi}$, which is $\left(-\frac{1}{3}\right)$ of the contribution of $\lambda(\mu)$

$$\chi = \frac{N_c}{3\pi^2} (e_q)^2 \sqrt{\frac{\mu_0}{2m_q}}, \quad (48)$$

which coincides with the total m.s. of the electron gas in the weak m.f., given in [25], when $m_q = m_e$, $N_c = 1$, $\mu_0 = \varepsilon_F$.

The same result can be obtained directly from (23) inserting there $L_\mu = \exp\left(\frac{\mu_0 + m_q}{T}\right)$, and using instead of x the variable $\frac{\varepsilon}{T} = \frac{1}{x} + \frac{m_q^2}{4T^2}x - \frac{m_q}{T}$. In the

limit $T \ll m_q$ one obtains $J_q \approx \sqrt{\frac{2}{m_q}} \int_0^\infty \frac{d\varepsilon}{\sqrt{\varepsilon}} \frac{1}{1+e^{\frac{\varepsilon-\mu_0}{T}}}$, which using the same technic as in (45) yields $\sqrt{\frac{2\mu_0}{m_q}}$ and one gets the result (48).

We now turn to the case b), $T \lesssim \frac{e_q B}{2m_q} \ll \mu_0$, which is interesting for us, since it provides the oscillating behavior, which is not present in our form (31), see discussion in § 59 of [25].

Indeed, the form (34) obtains, when one considers $e_q B$ outside of the interval b), or else, when one averages the result over some interval of $e_q B$, comprising many values of n_\perp in (31).

To this end we rewrite $\psi(\mu)$ in (31), separating the first term $\sigma = -1$, not depending on m.f. and use the Poisson formula

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) = \int_0^\infty F(x)dx + 2Re \sum_{k=1}^{\infty} \int_0^\infty dx F(x) e^{2\pi i k x}, \quad (49)$$

where

$$F(x) = \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \ln \left(1 + \exp \left(\frac{\bar{\mu} - \sqrt{p_z^2 + m_q^2 + 2e_q B x}}{T} \right) \right). \quad (50)$$

and $\bar{\mu} = m_q + \mu_0 - \frac{V_1(\infty, T)}{2}$ for quarks and $\bar{\mu} = m_e + \mu_0$ for the electron gas.

For small T , $\frac{e_q B}{m_q}$ as compared to m_q (nonrelativistic situation), we write the exponent as $\left(\mu_0 - \frac{p_z^2}{2m_q} - \frac{e_q B x}{m_q} \right) \frac{1}{T}$ and we are in the exact correspondence with the equations in § 60 of [25], when one replaces our μ_0, m_q, e_q by μ, m, e of the electron gas. The resulting expression for $P_q(B)$ (30) is

$$P_q(B) = N_c \frac{e_q B}{2\pi^2} (\phi(\mu) + \phi(-\mu)) - \frac{N_c T (e_q B)^{3/2}}{4\pi^2} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{2\pi \mu_0 m_q k}{e_q B} - \frac{\pi}{4} \right)}{k^{3/2} \text{sh} \left(\frac{2\pi^2 k T m_q}{e_q B} \right)}. \quad (51)$$

One can expect for the quark gas the same oscillations as in the de Haas-van Alphen effect, but the period of oscillations for quarks in $e_q B$ is $\mu_0 m_q$ and we assume $T \ll m_q$, hence this is improbable for a deconfined quark gas, where $T_c > m_q$, $q = u, d, s$. Therefore we shall try to proceed, assuming only that $T \lesssim \frac{e_q B}{2m_q}$, but allowing for $T \gg m_q$. Then the exponent in (50) can be rewritten as

$$\frac{\bar{\mu}^2 - m_q^2 - p_z^2 - 2e_q B x}{T(\bar{\mu} + \sqrt{m_q^2 + p_z^2 + 2e_q B x})} \equiv \frac{\delta\mu^2}{T} - \frac{(p_z^2 + 2e_q B x)}{2M_q T}, \quad (52)$$

where $\delta\mu = \frac{\bar{\mu}^2 - m_q^2}{2M_q}$ and $2M_q \equiv \bar{\mu} + \sqrt{m_q^2 + p_z^2 + 2e_q Bx}$.

Approximating M_q by some average value, not depending on p_z , one can exploit the final Eq. (51), replacing there m_q by M_q and M_0 by $\delta\mu$. As a result one expects the oscillations of $P_q(B)$ for growing $e_q B$ for $\frac{e_q B}{M_q T} \gtrsim 1$ and $T \ll \delta\mu$.

Indeed, the oscillating term in the integral (31), using (49), (50) can be written as (cf. § 60 of [25])

$$I_k = -e_q B \int_{-\infty}^{\infty} \int_0^{\infty} \ln \left[1 + \exp \left(\frac{\delta\mu}{T} - \frac{p_z^2 + 2e_q Bx}{2M_q T} \right) \right] e^{2\pi i k x} dp_z dx. \quad (53)$$

Introducing new variable $\varepsilon = \frac{p_z^2 + 2e_q Bx}{2M_q}$ instead of x , one obtains for the oscillating part of (53)

$$\bar{I}_k = - \int_{-\infty}^{\infty} \int_0^{\infty} \ln \left[1 + \exp \left(\frac{\delta\mu - \varepsilon}{T} \right) \right] \exp \left(\frac{2i\pi k \varepsilon M_q}{e_q B} \right) \exp \left(-\frac{i\pi k p_z^2}{e_q B} \right) d\varepsilon dp_z. \quad (54)$$

In (54) the essential part of integration region is $p_z^2 \sim e_q B$, while for the oscillating regime $\delta\mu \sim \varepsilon, \delta\mu \gg \frac{e_q B}{2M_q}$, therefore one replaces the lower limit of the ε integration by zero. Moreover, $2M_q \approx \bar{\mu} + \sqrt{m_q^2 + 2e_q Bx} \gtrsim \bar{\mu} \gg m_q$, and $\delta\mu \approx \bar{\mu}$.

Hence the final form of the oscillating part of the thermodynamic potential can be written instead of (51) as

$$\Delta P_q(B) = -N_c \frac{(e_q B)^{3/2} T}{4\pi^2} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{2\bar{\mu}^2}{e_q B} k - \frac{\pi}{4} \right)}{k^{3/2} \text{sh} \left(\frac{\pi^2 k T \bar{\mu}}{e_q B} \right)}. \quad (55)$$

One can see, that the relativistic quark gas potential (55) contains a much larger denominator due to $T \gg m_q$, as compared to the electron gas potential (51), leading to a relatively smaller amplitude of oscillations.

6 Summary and perspectives

We have developed above the theory of m.s. of the quark-antiquark matter in m.f., based on the explicit expressions for the thermodynamic potentials obtained by us in [15]. The case of m.s. for zero chemical potential μ was

studied in our previous paper [18], where it was shown, that m.s. $\hat{\chi}(T)$ is a strong function of T , growing with T due to Polyakov line factors. This behavior agrees well with recent lattice calculations [19, 20, 22], when one takes into account a possible modification of the effective quark mass as in (27). In the present paper we further examined the zero μ m.s., calculating m.s. for different quarks (u, d, s) and comparing with lattice data of [21] in our Fig. 2. As an additional topic we consider the free quark-antiquark gas m.s., which obtains from (23) putting $L_\mu \equiv 1$, and compare it with the corresponding m.s. of the electron gas from [30]. We observe a strong modification of the result due to the sum over Matsubara numbers. The main part of our results belong to the case of nonzero chemical potential μ in sections 4 and 5. Here m.s. $\hat{\chi}(T, \mu)$ has different behavior in the regions of small and large μ , $\mu \ll T$ and $\mu \gg T$. In the first case, considered in section 4, one can define the double magnetic- quark number susceptibilities. In the case of large μ , $\mu \gg T$ and $\frac{eB}{2m_q} \ll T$, one obtains the standard Pauli paramagnetism [23] and Landau diamagnetism [24] contributions to the m.s. given in (47).

Finally, in the case of large μ and large m.f. one arrives at the Landau theory of the de Haas-Van Alphen effect, written for nonrelativistic quarks in (50). The generalization to the case of relativistic quarks for $T \gg m_q$ is obtained in (54) and shows much milder amplitude of oscillations with growing $e_q B$. As it is we have developed the full theory of m.s. of the quark gas interacting with the QCD vacuum in the so-called Single Line Approximation (SLA) [37], when the interaction enters in the form of Polyakov lines. This allows to obtain m.s. at zero or small μ , and a good agreement was found with lattice data at least in the first case. In this approximation the interquark interactions are disregarded, however at larger μ (and hence larger quark densities) this effect can become important and this was discussed in [37, 40]. In SLA the QCD phase diagram in the $\mu - T$ plane was found in [15] and does not contain critical points. However for larger μ the interquark interaction becomes important and depends on μ both in the confined [44] and deconfined [39, 45] states. As a result the problem of the quark-hadron (qh) matter transitions should be solved with the full account of the interquark (beyond SLA) interactions. One aspect of this transition – the formation of the multiquark states and the nucleon matter was considered in [46], and shown to be important for the quark cores of neutron stars.

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